Numerical Dispersion Relation for FDTD Method in General Curvilinear Coordinates

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***Abstract*— The numerical dispersion relation (NDR) of the finite-difference time-domain method in general curvilinear coordinates (FDTD–GCC) is discussed for a two-dimensional (2-D) uniformly skewed mesh. The analysis shows that the average scheme, which is being used in the FDTD–GCC method, causes an additional numerical dispersion error. When this dispersion error is considered, the FDTD–GCC method holds the same NDR as that of the FDTD discrete surface integral (FDTD–DSI) method . It also indicates that the stable range of the FDTD–GCC method, with respect to the skewing angle in the 2-D case, is narrowed due to the average scheme.**

I. Introduction

Recently, the numerical dispersion relation (NDR) of the nonorthogonal finite-difference time-domain (FDTD) method has been reported by several authors [1]–[3]. Navarro *et al.* [1] have given out a general formula for the NDR of the FDTD method in general curvilinear coordinates (FDTD–GCC). By discretizing Ampere's and Faraday's laws, Ray [2] has provided a dispersion expression for a general nonorthogonal algorithm. The NDR given by Navarro *et al.* is slightly different from that provided by Ray. Navarro *et al.* said that it might be due to some mistake in [2]. Shi *et al.* [3] have developed a rigorous derivation of the NDR for the FDTD discrete surface integral (FDTD–DSI) technique. They addressed that the FDTD–GCC and the FDTD–DSI methods do not have the same NDR. The NDR presented by Navarro *et al.* [1] is correct for the FDTD–GCC method, while the NDR given by Ray [2] is applicable to the FDTD–DSI method.

This letter will discuss the NDR of the FDTD–GCC method in a two-dimensional (2-D) uniformly skewed mesh. The FDTD–GCC method uses the covariant and contravariant components of the electric and magnetic fields as the unknown variables. In the calculation procedure, the contravariant components of the fields must be computed from the covariant components, or vice versa. This requires all three of the corresponding components. However, these components are not known in the finite-difference cell at the same locations, therefore, an average scheme is employed [4]–[7]. In our analysis, the NDR of the FDTD–GCC method for a 2-D uniformly skewed mesh will be developed in two cases—the case of employing the average scheme and that without using this scheme. It will be shown that the average scheme causes an additional numerical dispersion error. Navarro's formula will be obtained only when the average scheme is not employed, while Ray's dispersion expression will be derived when the dispersion error caused by the average scheme is considered. Further, the stable range of the FDTD–GCC method, with respect to the skewing angle in the 2-D case, will be narrowed about 20° due to the average scheme.

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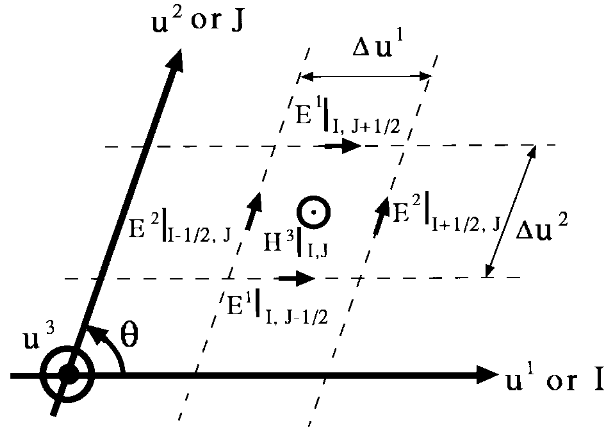

Fig. 1. Skewed mesh geometry and the field evaluation points on a finite-difference cell.

II. FDTD–GCC Algorithm and the Average Scheme

The general curvilinear coordinates system (u^1, u^2, u^3) may be characterized by the unitary vectors $\vec{a}_i$ or by the reciprocal unitary vectors $\vec{a}^i$ [8]. For example, the electric field $\vec{E}$ in the curvilinear space can be expanded as

$$\vec{E} = \sum_{i=1}^{3} E^i \vec{a}_i, \quad \vec{E} = \sum_{i=1}^{3} E_i \vec{a}^i \tag{1}$$

where the coefficients E^i and E_i are called the contravariant and covariant components of the electric field, respectively. The contravariant and covariant components are related by

$$E_i = \sum_{j=1}^{3} g_{ij} E^j, \quad E^i = \sum_{j=1}^{3} g^{ij} E_j \tag{2}$$

where $g_{ij} = \vec{a}_i \cdot \vec{a}_j$ and $g^{ij} = \vec{a}^i \cdot \vec{a}^j$.

We consider here the TE with respect to u^3 case [2] in a source-free uniform linear media shown in Fig. 1. In this figure, the positive u^2 axis is rotated an angle θ from the positive u^1 axis and the u^3 axis is perpendicular to the u^1u^2 plane. After the central difference approximation in time and space, Maxwell's equations are expressed as

$$H^3|_{I,J}^{n+1} - H^3|_{I,J}^{n} = -\frac{\Delta t}{\mu \sin\theta}\left(\frac{E_2|_{I+1/2,J}^{n+1/2} - E_2|_{I-1/2,J}^{n+1/2}}{\Delta u^1} - \frac{E_1|_{I,J+1/2}^{n+1/2} - E_1|_{I,J-1/2}^{n+1/2}}{\Delta u^2}\right) \quad (3)$$

$$E^1|_{I,J}^{n+1/2} - E^1|_{I,J}^{n-1/2} = \frac{\Delta t}{\varepsilon \sin\theta \Delta u^2} \cdot (H_3|_{I,J+1/2}^{n} - H_3|_{I,J-1/2}^{n}) \quad (4)$$

$$E^2|_{I,J}^{n+1/2} - E^2|_{I,J}^{n-1/2} = \frac{-\Delta t}{\varepsilon \sin\theta \Delta u^1} \cdot (H_3|_{I+1/2,J}^{n} - H_3|_{I-1/2,J}^{n}) \quad (5)$$

where the indices (I, J) and the increments Δu^1 and Δu^2 are referenced to Fig. 1. Note that on the right-hand sides of the above equations there appear the covariant components of the electric and magnetic fields, whereas the components on the left-hand sides are the contravariant components. Once the contravariant components of the fields are calculated, the covariant components must be computed by using (2). For the 2-D case, it follows from (2) that

$$E_1|_{I,J+1/2}^{n+1/2} = E^1|_{I,J+1/2}^{n+1/2} + \cos\theta E^2|_{I,J+1/2}^{n+1/2}. \quad (6)$$

Similarly, we can get the equations for the remaining components. From Fig. 1 we can see that the index for E^2 in (6) does not have the physical space meaning, because the contravariant components E^1 and E^2 are not known at the same locations. Equation (6) is written out just for the theoretical numerical dispersion analysis in the next section, and we denote this as the nonaverage case.

In the actual calculation, an average scheme that approximates the component value by averaging the four neighboring values is employed [4]–[7]. For the 2-D uniformly skewed meshes, it is in the form

$$E_1|_{I,J+1/2}^{n+1/2} = E^1|_{I,J+1/2}^{n+1/2} + \frac{1}{4}\cos\theta(E^2|_{I-1/2,J}^{n+1/2} + E^2|_{I+1/2,J}^{n+1/2} + E^2|_{I-1/2,J+1}^{n+1/2} + E^2|_{I+1/2,J+1}^{n+1/2}). \quad (7)$$

The equations of the remaining components can be obtained by straightforward permutation of indices.

III. Dispersion Analysis

The dispersion of the FDTD–GCC method is analyzed by assuming a plane wave propagating on the uniform skewed mesh. Let ω be its angular frequency, α be its propagation angle measured from the positive u^1 axis, and $k_1 = k\cos\alpha$ and $k_2 = k\cos(\alpha - \theta)$ be the u^1 and u^2 components of its numerical wavenumber k, respectively. We can assume

$$E^1|_{I,J}^{n+1/2} = E_0^1 e^{j[k_1 I \Delta u^1 + k_2 J \Delta u^2 - \omega(n+1/2)\Delta t]} \quad (8)$$

and the solutions for the remaining components. Using these trial solutions and substituting (6) into the difference equation (3) leads to

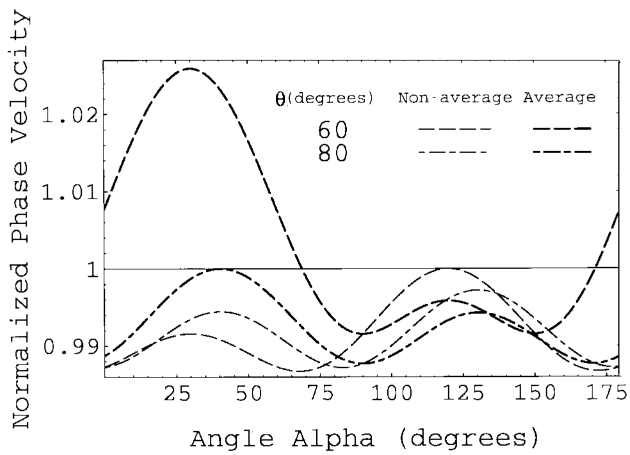

Fig. 2. Normalized phase velocity v_p/c versus propagation angle α in degree for various skewing angles θ of the grid with $\Delta u^1 = \Delta u^2 = 0.1\lambda_0$ and $c\Delta t = \Delta u^1/2$.

$$H_0^3 \sin(\frac{\omega\Delta t}{2}) = \frac{\Delta t}{\mu\sin\theta} \cdot \{\frac{1}{\Delta u^1}[E_0^2 \sin(\frac{k_1\Delta u^1}{2}) + E_0^1\cos\theta\sin(\frac{k_1\Delta u^1}{2})] - \frac{1}{\Delta u^2}[E_0^1\sin(\frac{k_2\Delta u^2}{2}) + E_0^2\cos\theta\sin(\frac{k_2\Delta u^2}{2})]\}. \quad (9)$$

Upon using (8) and the solutions of other components and introducing (4) and (5) into (9), we obtain the NDR of the FDTD–GCC method in the nonaverage case

$$\frac{\sin^2(\frac{1}{2}\omega\Delta t)}{(c\Delta t)^2} = \frac{1}{\sin^2\theta}\{\frac{1}{(\Delta u^1)^2}\sin^2(\frac{k\cos\alpha\Delta u^1}{2}) + \frac{1}{(\Delta u^2)^2}\sin^2[\frac{k\cos(\alpha-\theta)\Delta u^2}{2}] - \frac{2\cos\theta}{\Delta u^1\Delta u^2}\sin(\frac{k\cos\alpha\Delta u^1}{2}) \cdot \sin[\frac{k\cos(\alpha-\theta)\Delta u^2}{2}]\}. \quad (10)$$

Equation (10) is the same as the formula given by Navarro *et al.* [1].

In the same manner, using (7) we get the NDR of the FDTD–GCC method in the average case

$$\frac{\sin^2(\frac{1}{2}\omega\Delta t)}{(c\Delta t)^2} = \frac{1}{\sin^2\theta}\{\frac{1}{(\Delta u^1)^2}\sin^2(\frac{k\cos\alpha\Delta u^1}{2}) + \frac{1}{(\Delta u^2)^2}\sin^2[\frac{k\cos(\alpha-\theta)\Delta u^2}{2}] - \frac{\cos\theta}{2\Delta u^1\Delta u^2}\sin(k\cos\alpha\Delta u^1) \cdot \sin[k\cos(\alpha-\theta)\Delta u^2]\}. \quad (11)$$

Equation (11) is identical to that given by Ray [2] and Shi *et al.* [3]. For the TM case, the dispersion relations are obtained in a manner similar to that used for the TE case and are found to be the same as (10) and (11), respectively, for the nonaverage and average case. Some typical dispersion results by solving (10) and (11) are given in Fig. 2. In this figure, the grid spacing is chosen in terms of the wavelength as $\Delta u^1 = \Delta u^2 = 0.1\lambda_0$. It is shown that the FDTD–GCC method in the nonaverage case exhibits less dispersion than that of the average case. Furthermore, the FDTD–GCC method in the nonaverage case and the average case have different stable range with respect to the skewing angle θ. When $c\Delta t = \Delta u^1/2$ is maintained, the stable range of the former is $\theta \geq 60°$ while the stable range of the latter is about $\theta \geq 80°$, narrowed about $20°$.

IV. Conclusion

This letter has demonstrated that for a 2-D uniformly skewed mesh the FDTD–GCC and FDTD–DSI methods have the same NDR. The average scheme of the FDTD–GCC method also causes numerical dispersion error. Ray's dispersion expression, the NDR correct for the FDTD–DSI method, is also applicable to the FDTD–GCC method. The formula of Navarro *et al.* neglected the effect of the average scheme, therefore it is not applicable to the actual calculation case. Their formula is correct only if the three components of the fields are evaluated at the same locations. This will result in a marked increase in the number of field components and the computer resources needed, however. Another effect of the average scheme to the FDTD–GCC method is the shrink of the stable range. If we take the effect of the average scheme into account, the stability criterion for the FDTD–GCC method will differ from that given by [6]; this is being investigated presently.

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